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Research International Journal of Physics and Mathematical Sciences

Research Article

Bayesian Estimation of the Shape Parameter of Exponentiated lomax Distribution

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Received: 07 April, 2021
Accepted: 25 May, 2021
Published: 27 May, 2021

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Keywords: Bayesian method, exponentiated Lomax distribution, quasi and gamma priors, squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions.

Abstract

In this paper, the exponentiated Lomax distribution is considered for Bayesian analysis. The Bayes estimators of the shape parameter have been obtained under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors.

Introduction

The exponentiated Lomax distribution (ELD) has been proposed by Abdul-Moniem and Abdel [1]. They discussed some properties of this family. The probability density function $f(x;\theta)$, of ELD is given by

$$f(x;\theta) = a\lambda\theta[1+\lambda x]^{-(a+1)}\left[1-(1+\lambda x)^{-a}\right]^{\theta-1} \quad ; x>0. \tag{1}$$

The distribution function $F(x;\theta)$ and the survival function $S(x)$ of ELD are in the following forms

$$F(x;\theta) = \left[1-(1+\lambda x)^{-a}\right]^{\theta} \quad ; x>0. \tag{2}$$

$$S(x) = 1-\left[1-(1+\lambda x)^{-a}\right]^{\theta} \quad ; x>0. \tag{3}$$

The joint density function or likelihood function of (1) is given by

$$f(\underline{x};\theta) = (a\lambda\theta)^n\left(\prod_{i=1}^{n}[1+\lambda x_i]^{-(a+1)}\right)e^{-(\theta-1)\sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}} \tag{4}$$

Bayesian Method of Estimation

The Bayesian inference procedures have been developed generally under squared error loss function

$$L\left(\hat{\theta},\theta\right) = \left(\hat{\theta}-\theta\right)^2. \tag{5}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E(\theta). \tag{6}$$

Zellner [2], Basu and Ebrahimi [3] have recognized that the inappropriateness of using symmetric loss function. Norstrom [4] introduced precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}. \tag{7}$$

The Bayes estimator under this loss function is denoted by $\hat{\theta}_P$ and is obtained as

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2} \tag{8}$$

Calabria and Pulcini [5] point out that a useful asymmetric loss function is the entropy loss

$$L(\delta) \propto \left[\delta^p - p\ log_e(\delta) - 1\right]$$

where $\delta = \frac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta}=\theta$. Also, the loss function $L(\delta)$ has been used in Dey et al. [6] and Dey and Liu [7],

Citation: Rao A.K, Pandey H (2021) Bayesian Estimation of the Shape Parameter of Exponentiated lomax Distribution. Rea Int J of Phys and Math Sci.: 001-004. DOI: 10.37179/rijpms.000001

in the original form having $p=1$. Thus $L(\delta)$ can written be as

$$L(\delta)=b\left[\delta-\log_e(\delta)-1\right];\ b>0. \tag{9}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation

$$\hat{\theta}_E=\left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{10}$$

Wasan [8] proposed the K-loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K=\left[\frac{E(\theta)}{E(1/\theta)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [9] introduced a new loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\theta^c\left(\hat{\theta}-\theta\right)^2. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as

$$\hat{\theta}_{Al}=\frac{E\left(\theta^{c+1}\right)}{E\left(\theta^c\right)}. \tag{14}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta)=\frac{1}{\theta^d}\ ;\ \theta>0,\ d\geq 0, \tag{15}$$

where *d = 0* leads to a diffuse prior and *d = 1*, a non-informative prior.

(ii) Gamma prior: Generally, the gamma density is used as prior distribution of the parameter θ given by

$$g_2(\theta)=\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\ ;\ \theta>0. \tag{16}$$

Posterior Density under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (4), is given by

$$f(\theta/\underline{x})=\frac{(a\lambda\theta)^n\left(\prod_{i=1}^n[1+\lambda x_i]^{-(a+1)}\right)e^{-(\theta-1)\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}\theta^{-d}}{\int_0^\infty(a\lambda\theta)^n\left(\prod_{i=1}^n[1+\lambda x_i]^{-(a+1)}\right)e^{-(\theta-1)\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}\theta^{-d}d\theta}$$

$$=\frac{\theta^{n-d}\ e^{-\theta\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}}{\int_0^\infty\theta^{n-d}\ e^{-\theta\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}d\theta}$$

$$=\frac{\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)}\theta^{n-d}\ e^{-\theta\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}} \tag{17}$$

Theorem 1. On using (17), we have

$$E\left(\theta^c\right)=\frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-c}. \tag{18}$$

Proof. By definition,

$$E\left(\theta^c\right)=\int\theta^c f(\theta/\underline{x})d\theta$$

$$=\frac{\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)}\int_0^\infty\theta^{(n-d+c)}\ e^{-\theta\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}d\theta$$

$$=\frac{\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)}\frac{\Gamma(n-d+c+1)}{\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n-d+c+1}}$$

$$=\frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-c}.$$

From equation (18), for $c=1$, we have

$$E(\theta)=(n-d+1)\left(\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1}. \tag{19}$$

From equation (18), for $c=2$, we have

$$E\left(\theta^2\right)=\left[(n-d+2)(n-d+1)\right]\left[\sum_{i=1}^n\log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right]^{-2}. \tag{20}$$

From equation (18), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n-d)}\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1} \quad (21)$$

From equation (18), for $c=c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma(n-d+c+2)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-(c+1)} \quad (22)$$

Bayes Estimators under $g_1(\theta)$

From equation (6), on using (19), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=(n-d+1)\left(\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (23)$$

From equation (8), on using (20), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_P=\left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (24)$$

From equation (10), on using (21), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E=(n-d)\left(\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (25)$$

From equation (12), on using (19) and (21), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K=\left[(n-d+1)(n-d)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (26)$$

From equation (14), on using (18) and (22), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al}=(n-d+c+1)\left(\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (27)$$

Posterior Density under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (4), is obtained as

$$f(\theta/\underline{x})=\frac{(a\lambda\theta)^n\left(\prod_{i=1}^{n}[1+\lambda x_i]^{-(a+1)}\right)e^{-(\theta-1)\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}}{\int_0^{\infty}(a\lambda\theta)^n\left(\prod_{i=1}^{n}[1+\lambda x_i]^{-(a+1)}\right)e^{-(\theta-1)\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta}$$

$$=\frac{\theta^{n+\alpha-1}e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)\theta}}{\int_0^{\infty}\theta^{n+\alpha-1}e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)\theta}d\theta}$$

$$=\frac{\theta^{n+\alpha-1}e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)\theta}}{\Gamma(n+\alpha)\Big/\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n+\alpha}}$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)\theta} \quad (28)$$

Theorem 2. On using (28), we have

$$E\left(\theta^{c}\right)=\frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-c} \quad (29)$$

Proof. By definition,

$$E\left(\theta^{c}\right)=\int\theta^{c}f(\theta/\underline{x})d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^{\infty}\theta^{n+\alpha+c-1}e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)\theta}d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n+\alpha}}{\Gamma(n+\alpha)}\frac{\Gamma(n+\alpha+c)}{\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{n+\alpha+c}}$$

$$=\frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-c}.$$

From equation (29), for $c=1$, we have

$$E(\theta)=(n+\alpha)\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (30)$$

From equation (29), for $c=2$, we have

$$E\left(\theta^{2}\right)=\left[(n+\alpha+1)(n+\alpha)\right]\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-2} \quad (31)$$

From equation (29), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n+\alpha-1)}\left(\beta+\sum_{i=1}^{n}log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right) \quad (32)$$

From equation (29), for $c = c+1$, we have

$$E\left(\theta^{c+1}\right) = \frac{\Gamma(n+\alpha+c+1)}{\Gamma(n+\alpha)} \left(\beta + \sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-(c+1)} \quad (33)$$

Bayes Estimators under $g_2(\theta)$

From equation (6), on using (30), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S = (n+\alpha)\left(\beta + \sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (34)$$

From equation (8), on using (31), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_P = \left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}}\left(\beta + \sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (35)$$

From equation (10), on using (32), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = (n+\alpha+1)\left(\beta + \sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (36)$$

From equation (12), on using (30) and (32), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[(n+\alpha)(n+\alpha-1)\right]^{\frac{1}{2}}\left(\beta + \sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (37)$$

From equation (14), on using (29) and (33), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al} = (n+\alpha+c)\left(\beta + \sum_{i=1}^{n} log\left[1-(1+\lambda x_i)^{-a}\right]^{-1}\right)^{-1} \quad (38)$$

Conclusion

In this paper, we have obtained several estimators of parameter of exponentiated Lomax distribution. In equation (23), (24), (25), (26) and (27) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (34), (35), (36), (37) and (38) we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equation, the Bayes estimators depend upon the parameters of the prior distribution.

Acknowledgement

The authors are thankful to all referees for his valuable suggestion.

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